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# Central Limit Theorems for Motion-Invariant Poisson Hyperplanes in Expanding Convex Bodies

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## Abstract

It has been recently proved in Heinrich et al. [9] that the total number  $\Psi_0(B_\varrho^d)$  of intersection points generated by a stationary process of  $d$ -dimensional Poisson hyperplanes in a ball  $B_\varrho^d$  with radius  $\varrho$  is asymptotically normally distributed for large  $\varrho$ . In the present paper we generalize this and other results by replacing  $B_\varrho^d$  by an expanding sampling window  $\varrho K$ , where  $K$  is some fixed convex body with inner points. If the Poisson hyperplane process is additionally isotropic, the asymptotic variance of the scaled number  $\Psi_0(\varrho K)/\varrho^{d-1/2}$  of intersection points in  $\varrho K$  can be expressed in terms of a non-additive, motion-invariant ovoid functional of  $K$ , which is calculated explicitly for the unit ball  $B_1^d$ , ellipses and rectangles. Moreover, the method of  $U$ -statistics applied to the Poisson distributed number of hyperplanes hitting  $\varrho K$  allows to derive multivariate central limit theorems for the vector of numbers of intersection  $k$ -flats ( $k = 0, 1, \dots, d-1$ ) hitting  $\varrho K$  as well as for the vector of their total  $k$ -volumes ( $k = 0, 1, \dots, d-1$ ) within  $\varrho K$ .

**Keywords :** POISSON HYPERPLANE PROCESS,  $k$ -FLAT INTERSECTION PROCESS, HYPERPLANE TESSELLATION, INTRINSIC VOLUMES, CROFTON'S FORMULA,  $U$ -STATISTICS, NON-ADDITIVE OVOID FUNCTIONALS

**MCS 2000 :** 60D05, 60F05, 52A40, 53C65

## 1 Preliminaries and notation

In the last fifteen years central limit theorems (briefly CLTs) for geometric functionals and measures associated with random tessellations of the  $d$ -dimensional Euclidean space  $\mathbb{R}^d$  have been studied in various papers, see e.g. [5], [17], [18], [1], [8], [9], [10], [19]. The investigation of Gaussian limits for empirical characteristics such as the total length of cell boundaries or the number of vertices of the cells contained in expanding regions is of great relevance for the statistical analysis of random tessellations (construction of confidence intervals for mean-value estimators or asymptotic goodness-of-fit tests). Voronoi-tessellations as well as random line or plane tessellations are bench mark models to describe real-world cell structures which arise in diverse disciplines, for example in material sciences, see [15] or [16], and in modelling of telecommunication networks, see [8], [9]. For a lot of further applications of random tessellations and for the mathematical background from stochastic geometry and the theory of point processes we refer the reader to the monographs [16], [25], [12], [14], [23], and [4].

A CLT for the total number of vertices of convex polytopes induced by a stationary Poisson–Voronoi tessellation (briefly PVT) in cubes  $[-n, n]^d$  has been first proved in [5], see also [7] for calculating exact asymptotic variances of these numbers in case of  $d = 2$  and  $d = 3$ .

An analogous result for motion-invariant Poisson line tessellations in growing circular regions of the Euclidean plane was obtained by K. Paroux in [17] by employing the classical ‘method of moments’. In Heinrich et al. [9] Paroux’s CLT has been extended to stationary (not necessarily isotropic) Poisson hyperplane tessellations (briefly PHTs) in expanding ball in  $\mathbb{R}^d$  by applying Hoeffding’s decomposition for  $U$ -statistics with Poisson distributed unboundedly growing samples sizes.

It should be noted that the probabilistic properties of PVTs and PHTs are quite different. So stationary PVTs are absolutely regular (or  $\beta$ -mixing) with exponentially decaying mixing coefficient being responsible for Gaussian approximations of mean-value estimates, see [5], [10]. On the other hand, stationary PHTs have ‘long-range dependences’ which entail slowly decaying correlations between distant parts of PHTs. This is seen, for example, from the explicitly known ‘pair correlation function’ of the point process of vertices in motion-invariant PHTs, see e.g. [7]. For the same reason the variance of the total  $k$ -volume of  $k$ -facets ( $0 \leq k \leq d - 1$ ) of a stationary PHTs contained in a large (convex) sampling region of  $\mathbb{R}^d$  grows significantly faster than its  $d$ -volume and depends additionally on its shape. The strong mixing properties of stationary Voronoi tessellations ensure that the corresponding variances are proportional to the  $d$ -volume of the sampling region as it is known from many other classes of random set models, see e.g. [6].

In Section 2 we briefly introduce some notions from integral and stochastic geometry needed in the sequel and recall the definition of a stationary (isotropic) Poisson hyperplane process (briefly PHPs) in  $\mathbb{R}^d$ . In Section 3 we consider motion-invariant PHPs in an expanding convex region  $\varrho K$ , where  $K$  is a convex body containing the origin as inner point. We prove a CLT for the joint distribution of the number of intersection points and the number of intersection  $k$ -flats ( $k = 1, \dots, d - 1$ ) hitting  $\varrho K$  as  $\varrho \rightarrow \infty$ . Section 4 is focused on proving Gaussian limits for the  $k$ -dimensional Lebesgue measure of the union of intersection  $k$ -flats (generated by intersection of  $d - k$  hyperplanes) in  $\varrho K$  for  $k = 0, 1, \dots, d - 1$ . Note that for studying these random total  $k$ -volumes the convexity of  $K$  can be weakened. It is also noteworthy that the rank of the corresponding asymptotic covariance matrix equals 1 in any dimension  $d \geq 2$ . Moreover, we show in Section 5 that the covariance matrix of the Gaussian limit in Section 2 has full rank for any  $d \geq 2$ . However, the entries of this covariance matrix strongly depend on the shape of  $K$ . In particular the asymptotic variance of each component is up to some constant a non-additive, motion-invariant ovoid functional of  $K$  which seems to be of interest in its own right. In Section 6 we give some estimates of these functionals and derive explicit formulas in case of balls ( $d \geq 2$ ) and for ellipses and rectangles ( $d = 2$ ).

At the end of this section we introduce some basic notation used repeatedly in this paper. Throughout, let  $\mathfrak{B}^d$  be the  $\sigma$ -algebra of Borel sets in  $\mathbb{R}^d$  and let  $[\Omega, \mathfrak{A}, \mathbf{P}]$  be a common probability space on which all random objects are defined in this paper. The symbols  $\mathbf{E}$ ,  $\mathbf{Var}$  and  $\mathbf{Cov}$  are used for expectation, variance and covariance w.r.t. the probability measure  $\mathbf{P}$ . Further, let  $\langle x, y \rangle = \sum_{k=1}^d x_k y_k$  denote the scalar product of the coordinate vectors  $x = (x_1, \dots, x_d)^\top$  and  $y = (y_1, \dots, y_d)^\top$  in  $\mathbb{R}^d$ . By means of the Euclidean norm

$\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$  we may define the closed ball  $B_r^d = \{x \in \mathbb{R}^d : \|x\| \leq r\}$  with radius  $r \geq 0$  centered at the origin and the unit sphere  $\mathbf{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\| = 1\}$  in  $\mathbb{R}^d$ , respectively. Furthermore, let  $\mathbf{S}_+^{d-1} = \{(x_1, \dots, x_d)^\top \in \mathbf{S}^{d-1} : x_d \geq 0\}$  be the upper unit hemisphere and let  $\nu_k(\cdot)$  denote the Lebesgue measure in  $\mathbb{R}^k$  for  $k = 0, 1, \dots, d$ . This measure will also be used instead of the  $k$ -dimensional Hausdorff measure in  $\mathbb{R}^d$  for  $k = 0, \dots, d-1$ . As usual,  $\nu_0(\cdot)$  coincides with the counting measure, i.e.,  $\nu_0(B) = \#B$ . The  $d$ -volume of the unit ball is abbreviated by  $\kappa_d = \nu_d(B_1^d)$  with

$$\kappa_{2k} = \frac{\pi^k}{k!} \quad \text{and} \quad \kappa_{2k+1} = \frac{2^{2k+1} k! \pi^k}{(2k+1)!} \quad \text{for } k = 0, 1, \dots$$

## 2 Stationary Poisson hyperplane processes in $\mathbb{R}^d$ and their $k$ -flat intersection processes, $0 \leq k \leq d-1$

An unoriented hyperplane in  $\mathbb{R}^d$  can be represented in the parametrized form

$$H(p, v) = \{x \in \mathbb{R}^d : \langle x, v \rangle = p\}$$

with *orientation vector*  $v \in \mathbf{S}_+^{d-1}$  (the normal unit vector of  $H(p, v)$ ) and  $p \in \mathbb{R}^1$  gives the signed perpendicular distance from the origin.

**Definition** A stationary *Poisson hyperplane process*  $\Phi_{\lambda, \Theta}^{(d)}$  in  $\mathbb{R}^d$  is defined to be a countable family  $\{H(P_i, V_i) : i \geq 1\}$  of random  $(d-1)$ -dimensional affine linear subspaces of  $\mathbb{R}^d$  (hyperplanes), where  $\Psi = \{[P_i, V_i] : i \geq 1\}$  is a stationary independently marked Poisson point process on the real line  $\mathbb{R}^1$  with intensity  $0 < \lambda < \infty$  and mark distribution  $\Theta(\cdot)$  (called *orientation distribution* of the PHP) given on the measurable mark space  $[\mathbf{S}_+^{d-1}, \mathfrak{B}^d \cap \mathbf{S}_+^{d-1}]$ .

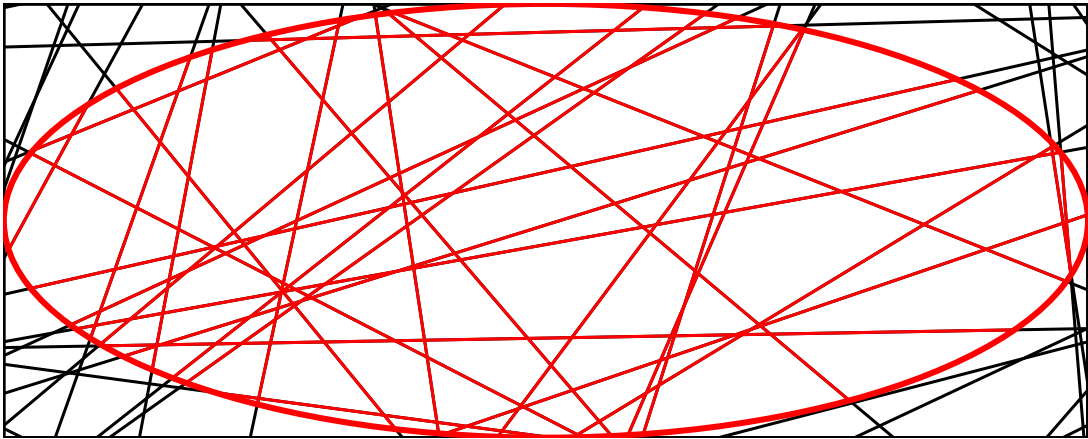


Figure 1: Motion-invariant Poisson line process in a rectangular and elliptic window

Note that non-stationary PHPs can be defined in the same way by letting  $\Psi$  an independently marked Poisson process with an intensity measure  $\Lambda(\cdot)$  on  $[\mathbb{R}^1, \mathfrak{B}^1]$  being not shift-invariant, see e.g. [24]. A stationary PHP  $\Phi_{\lambda, \Theta}^{(d)}$  (or its orientation distribution  $\Theta(\cdot)$ ) is called *non-degenerate* if  $\Theta(H(0, v) \cap \mathbf{S}_+^{d-1}) < 1$  for all  $v \in \mathbf{S}_+^{d-1}$ . This assumption on  $\Theta(\cdot)$  ensures that each of the stationary  $k$ -flat intersection processes

$$\Phi_{\lambda, \Theta}^{(d, k)} = \{H(P_{i_1}, V_{i_1}) \cap \cdots \cap H(P_{i_{d-k}}, V_{i_{d-k}}) : 1 \leq i_1 < \cdots < i_{d-k}\}$$

has positive mean  $k$ -volume in the unit cube  $[0, 1]^d$  for each  $k = 0, 1, \dots, d-1$  and the stationary PHT induced by  $\Phi_{\lambda, \Theta}^{(d)}$  consists of bounded cells; see Chapter 6 in [23]. In the latter reference the reader can find a rigorous introduction into the general theory of stationary  $k$ -flat processes in  $\mathbb{R}^d$  which are described there as point processes on the space  $\mathcal{A}_k^d$  of all  $k$ -dimensional affine subspaces in  $\mathbb{R}^d$ ,  $0 \leq k \leq d-1$ . A  $(d-1)$ -flat process is usually called hyperplane process. Sometimes the orientation distribution of a PHP is introduced as an even probability measure  $\Theta^*(\cdot)$  on the entire sphere  $\mathbf{S}^{d-1}$  which is then connected with  $\Theta(\cdot)$  by  $\Theta^*(B) = \frac{1}{2} \left( \Theta(B \cap \mathbf{S}_+^{d-1}) + \Theta(-B \cap \mathbf{S}_+^{d-1}) \right)$  for any Borel set  $B \subseteq \mathbf{S}^{d-1}$  and, conversely,  $\Theta(B) = \Theta^*(B) + \Theta^*(-B)$  for any Borel set  $B \subseteq \mathbf{S}_+^{d-1}$ . The symmetry condition  $\Theta^*(B) = \Theta^*(-B)$  expresses the identification of hyperplanes with antipodal orientation vectors. A stationary PHP  $\Phi_{\lambda, \Theta}^{(d)}$  is said to be *isotropic* (or *motion-invariant*) if  $\Theta(\cdot)$  is the uniform distribution  $U(\cdot) = 2\nu_{d-1}(\cdot)/d\kappa_d$  on  $\mathbf{S}_+^{d-1}$ , i.e.  $\Theta^*(\cdot)$  is the uniform distribution on  $\mathbf{S}^{d-1}$ . It turns out that the union sets

$$\Xi_{\lambda, \Theta}^{(d, k)} = \bigcup_{1 \leq i_1 < \cdots < i_{d-k}} \bigcap_{j=1}^{d-k} H(P_{i_j}, V_{i_j}) \quad \text{for } 0 \leq k \leq d-1$$

are stationary resp. motion-invariant random closed sets, see [12], iff the PHP  $\Phi_{\lambda, \Theta}^{(d)}$  does so in the above-defined sense. The intensity  $\lambda$  of  $\Phi_{\lambda, \Theta}^{(d)}$  can be expressed by

$$\lambda = \frac{1}{2r} \mathbb{E} \# \{i \geq 1 : H(P_i, V_i) \cap B_r^d \neq \emptyset\} = \frac{1}{\nu_d(B)} \mathbb{E} \nu_{(d, d-1)}(\Xi_{\lambda, \Theta}^{(d)} \cap B)$$

for any  $r > 0$  and any bounded  $B \in \mathfrak{B}^d$ .

Now we are in a position to introduce two functionals  $\Psi_k^{(d)}(\cdot)$  and  $\zeta_k^{(d)}(\cdot)$  on the family of sets  $K_\varrho := \varrho K$ ,  $\varrho \geq 1$ , where  $K$  is convex and compact such that  $B_\varepsilon^d \subseteq K$  for some  $\varepsilon > 0$ . For  $k = 0, 1, \dots, d-1$  define

$$\Psi_k^{(d)}(K_\varrho) = \frac{1}{(d-k)!} \sum_{i_1, \dots, i_{d-k} \geq 1}^* \chi\left(\bigcap_{j=1}^{d-k} H(P_{i_j}, V_{i_j}) \cap K_\varrho\right) \quad (2.1)$$

and

$$\zeta_k^{(d)}(K_\varrho) = \frac{1}{(d-k)!} \sum_{i_1, \dots, i_{d-k} \geq 1}^* \nu_k\left(\bigcap_{j=1}^{d-k} H(P_{i_j}, V_{i_j}) \cap K_\varrho\right), \quad (2.2)$$

where  $\chi(C) = 1$  for  $C \neq \emptyset$  and  $\chi(\emptyset) = 0$  and the asterisk in  $\sum^*$  indicates that the sum runs over pairwise distinct indices  $i_1, \dots, i_{d-k} \geq 1$ . Obviously,  $\Psi_k^{(d)}(K_\varrho)$  counts

the number of intersection  $k$ -flats of  $\Phi_{\lambda, \Theta}^{(d)}$  hitting  $K_\varrho$ , whereas  $\zeta_k^{(d)}(K_\varrho)$  measures their the total  $k$ -volume in  $K_\varrho$ .

To determine expectations, variances and last but not least the asymptotic behaviour (as  $\varrho \rightarrow \infty$ ) of (2.1) and (2.2) we consider the stationary independently marked Poisson process  $\Psi$  on  $\mathbb{R}^1$  as a non-stationary Poisson process on the product space  $\mathbb{R}^1 \times \mathbf{S}_+^{d-1}$ . An equivalent formulation of the Poisson property of this point process enables us to argue as follows: Given the Poisson distributed number

$$N(K_\varrho) = \#\{i \geq 1 : H(P_i, V_i) \cap K_\varrho \neq \emptyset\} = \sum_{i \geq 1} \chi(H(P_i, V_i) \cap K_\varrho)$$

of hyperplanes hitting  $K_\varrho$ , say  $N(K_\varrho) = n$ , the random vectors  $X_i^{(\varrho)} = (P_i, V_i)$ ,  $i = 1, \dots, n$ , are (conditionally) independent of  $N(K_\varrho)$  and independent identically distributed (briefly IID) with common distribution  $Q_\Theta^{(\varrho)}(\cdot)$  which is defined, for  $B \in \mathfrak{B}^1$  and  $S \in \mathfrak{B}^d \cap \mathbf{S}_+^{d-1}$ , by

$$\begin{aligned} Q_\Theta^{(\varrho)}(B \times S) &= \frac{1}{b_\Theta(K_\varrho)} \int_B \int_S \chi(H(p, v) \cap K_\varrho) \Theta(dv) dp \\ &= \frac{1}{b_\Theta(K)} \int_B \int_S \chi(H(\varrho^{-1}p, v) \cap K) \Theta(dv) dp, \end{aligned}$$

where

$$b_\Theta(K) = \int_{\mathbb{R}^1} \int_{\mathbf{S}_+^{d-1}} \chi(H(p, v) \cap K) \Theta(dv) dp. \quad (2.3)$$

For brevity put  $X_i = X_i^{(1)}$  for  $i = 1, 2, \dots$  and let  $X_0$  denote the generic random vector with distribution  $Q_\Theta^{(1)}(\cdot)$ . Therefore we may write

$$\Psi_k^{(d)}(K_\varrho) \stackrel{d}{=} \sum_{1 \leq i_1 < \dots < i_{d-k} \leq N(K_\varrho)} \chi\left(\bigcap_{j=1}^{d-k} H(X_{i_j}^{(\varrho)}) \cap K_\varrho\right) \quad (2.4)$$

$$\zeta_k^{(d)}(K_\varrho) \stackrel{d}{=} \sum_{1 \leq i_1 < \dots < i_{d-k} \leq N(K_\varrho)} \nu_k\left(\bigcap_{j=1}^{d-k} H(X_{i_j}^{(\varrho)}) \cap K_\varrho\right). \quad (2.5)$$

Here the symbol  $\stackrel{d}{=}$  indicates distributional equality. The expectation of  $N(K_\varrho)$  can be calculated by means of Campbell's theorem, see e.g. [4], and leads to the following simple formula:

$$\mathbb{E} N(K_\varrho) = \mathbb{E} \sum_{i \geq 1} \chi(H(P_i, V_i) \cap K_\varrho) = \lambda \varrho b_\Theta(K),$$

The integral  $b_\Theta(K)$  defined in (2.3) can be rewritten as

$$b_\Theta(K) = \int_{\mathbb{S}_+^{d-1}} (h_K(v) + h_K(-v)) \Theta(dv),$$

where  $h_K(v) = \max_{x \in K} \langle v, x \rangle$  denotes the *support function* of the convex body  $K$ . The last relation becomes clear since  $h_K(v)$  equals just the distance of the support hyperplane with exterior normal unit vector  $u$  from the origin. The value of  $b_\Theta(K)$  can be interpreted as ‘direction-weighted breadth’ of the convex body  $K$  w.r.t. the distribution  $\Theta(\cdot)$ .

If the PHP  $\Phi_{\lambda, \Theta}^{(d)}$  is motion-invariant, i.e.  $\Theta(\cdot) = U(\cdot)$ , then  $b_U(K)$  coincides with the *mean breadth*  $b(K)$  of  $K$  which is a well-known from integral geometry, see [22].

In what follows we put special emphasis on motion-invariant PHPs  $\Phi_{\lambda, U}^{(d)}$ . In this case the calculation of (conditional) expectations will be done by iterated application of *Crofton’s formula* which can be written as follows:

$$\int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^1} V_k^{(d)}(H(p, v) \cap K) dp U(dv) = \frac{\kappa_{d-1}}{d \kappa_d} \frac{(k+1) \kappa_{k+1}}{\kappa_k} V_{k+1}^{(d)}(K) \quad (2.6)$$

for  $k = 0, 1, \dots, d-1$ , where  $V_k^{(d)}(K)$  is called the  $k$ -th *intrinsic volume* of the convex body  $K \subset \mathbb{R}^d$ , see [22]. These additive, motion-invariant and increasing ovoid functionals can be defined by the well-known *Steiner formula*

$$\nu_d(K + B_r^d) = \sum_{k=0}^d r^{d-k} \kappa_{d-k} V_k^{(d)}(K) \quad \text{for } r \geq 0.$$

The intrinsic volumes of  $K$  are connected with *Minkowski’s quermassintegrals*  $W_k^{(d)}(K)$  by  $\binom{d}{k} W_k^{(d)}(K) = \kappa_k V_{d-k}^{(d)}(K)$  for  $k = 0, 1, \dots, d-1$ . We list some special cases:

$$V_0^{(d)}(K) = \chi(K), \quad V_1^{(d)}(K) = \frac{d \kappa_d}{2 \kappa_{d-1}} b(K), \quad V_{d-1}^{(d)}(K) = \frac{1}{2} \nu_{d-1}(\partial K), \quad V_d^{(d)}(K) = \nu_d(K)$$

and  $V_k^{(d)}(K) = \nu_k(K)$  if  $V_{k+1}^{(d)}(K) = 0$  for some  $k \leq d-1$ .

Next we calculate the mean values of (2.1) and (2.2). For doing this we use the distributional identities (2.4) and (2.5) and the well-known fact that the  $(d-k)$ -th factorial moment of the Poisson distributed number  $N(K_\varrho)$  equals  $(\mathbb{E}N(K_\varrho))^{d-k} = (\lambda \varrho b_\Theta(K))^{d-k}$  so that, for a stationary PHP,

$$\mathbb{E} \Psi_k^{(d)}(K_\varrho) = \mathbb{E} \binom{N(K_\varrho)}{d-k} \mathbb{E} \chi \left( \bigcap_{i=1}^{d-k} H(X_i^{(\varrho)}) \cap K_\varrho \right) = (\lambda \varrho)^{d-k} \mu_\Theta^{(k,d)}(K), \quad (2.7)$$

where

$$\mu_\Theta^{(k,d)}(K) := \frac{(b_\Theta(K))^{d-k}}{(d-k)!} \mathbb{P} \left( \bigcap_{i=1}^{d-k} H(X_i) \cap K \neq \emptyset \right)$$



$$= \frac{1}{(d-k)!} \int_{\mathbf{S}_+^{d-1}} \int_{\mathbf{R}^1} \cdots \int_{\mathbf{S}_+^{d-1}} \int_{\mathbf{R}^1} \chi\left(\bigcap_{i=1}^{d-k} H(p_i, v_i) \cap K\right) dp_1 \Theta(dv_1) \cdots dp_{d-k} \Theta(dv_{d-k}) .$$

If  $\Theta(\cdot) = U(\cdot)$  we may apply Crofton's formula (2.6)  $d-k$  times which leads to

$$\mu_U^{(k,d)}(K) = \prod_{j=0}^{d-k-1} \left( \frac{\kappa_{d-1}}{d \kappa_d} \frac{(j+1) \kappa_{j+1}}{\kappa_j} \right) \frac{V_{d-k}^{(d)}(K)}{(d-k)!} = \left( \frac{\kappa_{d-1}}{d \kappa_d} \right)^{d-k} \kappa_{d-k} V_{d-k}^{(d)}(K) . \quad (2.8)$$

In the same manner we find that

$$\mathbb{E} \zeta_k^{(d)}(K_\varrho) = \frac{(\lambda \varrho b_\Theta(K))^{d-k}}{(d-k)!} \varrho^k \mathbb{E} \nu_k\left(\bigcap_{i=1}^{d-k} H(X_i) \cap K\right) = \lambda^{d-k} \lambda_\Theta^{(k,d)} \varrho^d \nu_d(K) , \quad (2.9)$$

where

$$\lambda_\Theta^{(k,d)} = \frac{1}{(d-k)!} \int_{\mathbf{S}_+^{d-1}} \int_{\mathbf{R}^1} \cdots \int_{\mathbf{S}_+^{d-1}} \int_{\mathbf{R}^1} \nu_k\left(\bigcap_{i=1}^{d-k} H(p_i, v_i) \cap B_0\right) dp_1 \Theta(dv_1) \cdots dp_{d-k} \Theta(dv_{d-k}) .$$

Here,  $B_0$  can be any bounded Borel set in  $\mathbb{R}^d$  satisfying  $\nu_d(B_0) = 1$ . Again by multiple application of (2.6) it follows that

$$\lambda_U^{(k,d)} = \binom{d}{k} \frac{\kappa_d}{\kappa_k} \left( \frac{\kappa_{d-1}}{d \kappa_d} \right)^{d-k} . \quad (2.10)$$

Note that the relation (2.9) remains valid for any bounded  $K \in \mathfrak{B}^d$  (the term  $b_\Theta(K)$  cancels out) so that  $\mathbb{E} \zeta_k^{(d)}(B_0) = \lambda^{d-k} \lambda_\Theta^{(k,d)}$  can be regarded as intensity of the random measure  $\zeta_k^{(d)}(\cdot)$  as well as of the  $k$ -flat intersection process  $\Psi_{\lambda, \Theta}^{(k,d)}$ . Furthermore, one can rewrite  $\lambda_\Theta^{(k,d)}$  as follows:

$$\lambda_\Theta^{(k,d)} = \frac{1}{(d-k)!} \int_{\mathbf{S}_+^{d-1}} \cdots \int_{\mathbf{S}_+^{d-1}} \nabla_{d-k}(v_1, \dots, v_{d-k}) \Theta(dv_1) \cdots \Theta(dv_{d-k}) = V_{d-k}^{(d)}(Z_\Theta)$$

for  $k = 0, 1, \dots, d-1$ , where  $\nabla_{d-k}(v_1, \dots, v_{d-k})$  denotes the  $(d-k)$ -dimensional volume of the parallelotope spanned by  $v_1, \dots, v_{d-k} \in \mathbf{S}_+^{d-1}$  and  $Z_\Theta$  is a centrally symmetric convex body uniquely determined by the orientation distribution  $\Theta(\cdot)$  - called *associated zonoid* or *Steiner compact* in [12], see [23] for details.

At the end of this section we recall some basic facts on U-statistics. Let  $Y_1, Y_2, \dots$  be a sequence of IID random elements in some measurable space  $[E, \mathfrak{E}]$  and, for fixed  $m \geq 2$ ,

let  $f : E^m \rightarrow \mathbb{R}^1$  be a  $\mathfrak{E}^m$ -measurable symmetric function such that  $\mathbb{E}|f(Y_1, \dots, Y_m)| < \infty$ . A  $U$ -statistic  $U_n^{(m)}(f)$  of order  $m \geq 1$  with *kernel function*  $f$  is defined by

$$U_n^{(m)}(f) = \sum_{1 \leq i_1 < \dots < i_m \leq n} f(Y_{i_1}, \dots, Y_{i_m}) \quad \text{for } n \geq m.$$

The representations (2.4) and (2.5) reveal that  $\Psi_k^{(d)}(K_\varrho)$  as well as  $\zeta_k^{(d)}(K_\varrho)$  can be treated as random  $U$ -statistics of order  $d - k$  with Poisson distributed sample size  $N((K_\varrho))$ . In contrast to the standard case the kernel function  $\chi(\bigcap_{i=1}^{d-k} H(y_i) \cap K_\varrho)$  resp.  $\nu_k(\bigcap_{i=1}^{d-k} H(y_i) \cap K_\varrho)$  and the common distribution  $Q_\Theta^{(\varrho)}(\cdot)$  of the  $Y_i$ 's depend on the (mean) sample size.

The proofs of the CLTs we are going to present in Sections 3 and Sections 4 rely on Hoeffding's decomposition of  $U$ -statistics which allows to approximate  $U_n^{(m)}(f)$  by a simple sum of independent random variables :

$$U_n^{(m)}(f) - \binom{n}{m} \mu = \binom{n-1}{m-1} \sum_{i=1}^n (g(Y_i) - \mu) + \binom{n}{m} R_n^{(m)}(f), \quad (2.11)$$

where  $\mu = \mathbb{E}f(Y_1, Y_2, \dots, Y_m)$  and  $g(y) = \mathbb{E}f_m(y, Y_2, \dots, Y_m)$  is the conditional expectation of  $f(Y_1, Y_2, \dots, Y_m)$  given  $Y_1 = y \in E$ . The remainder term  $R_n^{(m)}(f)$  contains sums of martingale difference sequences arising in Hoeffding's decomposition which in turn allows to prove the following crucial estimate given in

**Lemma 2.1** *Provided that  $\mathbb{E}f^2(Y_1, \dots, Y_m) < \infty$  we have*

$$\mathbb{E}(R_n^{(m)}(f))^2 \leq \frac{c_m}{n^2} \mathbb{E}f^2(Y_1, \dots, Y_m) \quad \text{for any } n \geq m$$

*and some positive constant  $c_m$  only depending on the order  $m$ .*

This result is the main step in the proof of Hoeffding's CLT for  $U$ -statistics, see e.g. [3] or [9] for a sketch. In [3] the reader can find a lot of further details on  $U$ -statistics including weaker versions of Lemma 2.1.

### 3 CLTs for the number of intersection $k$ -flats hitting $\varrho K$

In this section we study the joint asymptotic behaviour (as  $\varrho \rightarrow \infty$ ) of the centered and scaled random variables

$$\overline{\Psi}_{k,\varrho}^{(d)}(K) = \varrho^{-(d-k-1/2)} \left( \Psi_k^{(d)}(K_\varrho) - (\lambda \varrho)^{d-k} \mu_\Theta^{(k,d)}(K) \right) \quad (3.1)$$

for  $k = 0, 1, \dots, d-1$  in case of a stationary PHP  $\Psi_{\lambda,\Theta}^{(d)}$  with  $\mu_\Theta^{(k,d)}(K)$  defined in (2.7). More precisely, we shall prove a multivariate CLT for the random vector  $(\overline{\Psi}_{k,\varrho}^{(d)}(K))_{k=0}^{d-1}$

and determine the covariance matrix of the Gaussian limiting vector in the particular case  $\Theta(\cdot) = U(\cdot)$ . To use results from the theory of  $U$ -statistics we introduce the conditional expectation

$$g_{\chi, \Theta}^{(k,d)}((p, v), K) = \mathbb{E} \chi \left( \bigcap_{i=1}^{d-k-1} H(X_i) \cap H(p, v) \cap K \right)$$

for  $(p, v) \in \mathbb{R}^1 \times \mathbf{S}_+^{d-1}$ . In analogy to (2.8) we apply Crofton's formula (2.6)  $d-k-1$  times (with  $K$  replaced by  $H(p, v) \cap K$ ) to derive a more explicit formula of the conditional expectation  $g_{\chi, U}^{(k,d)}((p, v), K)$ , namely

$$g_{\chi, U}^{(k,d)}((p, v), K) = a_k^{(d)} \frac{(d-k-1)!}{(b(K))^{d-k-1}} V_{d-k-1}^{(d)}(H(p, v) \cap K) \quad (3.2)$$

with

$$a_k^{(d)} = \left( \frac{\kappa_{d-1}}{d \kappa_d} \right)^{d-k-1} \kappa_{d-k-1} \quad \text{for } k = 0, 1, \dots, d-1.$$

In the first step we calculate the asymptotic covariances

$$\sigma_{kl}^{(d)}(\Theta, K) = \lim_{\varrho \rightarrow \infty} \text{Cov}(\overline{\Psi}_{k, \varrho}^{(d)}(K), \overline{\Psi}_{l, \varrho}^{(d)}(K)).$$

For brevity, put  $\sigma_{kl}^{(d)}(K) = \sigma_{kl}^{(d)}(U, K)$  for  $k, l = 0, 1, \dots, d-1$ .

**Lemma 3.1** *Let  $\Psi_{\lambda, \Theta}^{(d)}$  be a stationary PHP with non-degenerate orientation distribution  $\Theta(\cdot)$ . Then,*

$$\sigma_{kl}^{(d)}(\Theta, K) = \frac{(\lambda b_{\Theta}(K))^{2d-k-l-1}}{(d-k-1)!(d-l-1)!} \mathbb{E} g_{\chi, \Theta}^{(k,d)}(X_0, K) g_{\chi, \Theta}^{(l,d)}(X_0, K) > 0 \quad (3.3)$$

for  $k, l = 0, 1, \dots, d-1$ .

In the particular case of a motion-invariant PHP  $\Psi_{\lambda, U}^{(d)}$ , we have

$$\begin{aligned} \sigma_{kl}^{(d)}(K) &= \lambda^{2d-k-l-1} a_k^{(d)} a_l^{(d)} \\ &\times \int_{\mathbf{S}_+^{d-1}} \int_{\mathbb{R}^1} V_{d-k-1}^{(d)}(H(p, v) \cap K) V_{d-l-1}^{(d)}(H(p, v) \cap K) \, dp \, U(dv). \end{aligned} \quad (3.4)$$

**Proof** By symmetry of the kernel function  $\chi(H(y_1) \cap \dots \cap H(y_m) \cap K_{\varrho})$  for  $m = d-k$  and  $m = d-l$  we may write

$$(d-k)!(d-l)! \mathbb{E}(\Psi_k^{(d)}(K_{\varrho}) \Psi_l^{(d)}(K_{\varrho}))$$

$$= \mathbb{E} \left( \sum_{1 \leq i_1, \dots, i_{d-k} \leq N(K_\varrho)}^* \chi \left( \bigcap_{p=1}^{d-k} H(X_{i_p}^{(\varrho)}) \cap K_\varrho \right) \right) \left( \sum_{1 \leq j_1, \dots, j_{d-l} \leq N(K_\varrho)}^* \chi \left( \bigcap_{q=1}^{d-l} H(X_{j_q}^{(\varrho)}) \cap K_\varrho \right) \right)$$

Some basic combinatorial relations combined with the independence assumptions made in (2.4) yield that, for  $0 \leq k \leq l \leq d-1$ ,

$$\begin{aligned} \mathbb{E}(\Psi_k^{(d)}(K_\varrho) \Psi_l^{(d)}(K_\varrho)) &= \sum_{j=0}^{d-l} \frac{j!}{(d-k)!(d-l)!} \binom{d-k}{j} \binom{d-l}{j} \\ &\times \mathbb{E} \left( \sum_{1 \leq i_1, \dots, i_{2d-k-l-j} \leq N(K_\varrho)}^* \chi \left( \bigcap_{p=1}^{d-k} H(X_{i_p}^{(\varrho)}) \cap K_\varrho \right) \chi \left( \bigcap_{q=d-k-j+1}^{2d-k-l-j} H(X_{i_q}^{(\varrho)}) \cap K_\varrho \right) \right) \\ &= \sum_{j=0}^{d-l} \frac{j! (2d-k-l-j)!}{(d-k)!(d-l)!} \binom{d-k}{j} \binom{d-l}{j} \mathbb{E} \left( \binom{N(K_\varrho)}{2d-k-l-j} \right) \\ &\times \mathbb{E} \left( \chi \left( \bigcap_{p=1}^{d-k} H(X_p^{(\varrho)}) \cap K_\varrho \right) \chi \left( \bigcap_{q=d-k-j+1}^{2d-k-l-j} H(X_q^{(\varrho)}) \cap K_\varrho \right) \right) \\ &= \mathbb{E} \Psi_k^{(d)}(K_\varrho) \mathbb{E} \Psi_l^{(d)}(K_\varrho) + \sum_{j=1}^{d-l} \frac{(\lambda b_\Theta(K) \varrho)^{2d-k-l-j}}{j! (d-k-j)! (d-l-j)!} \\ &\times \mathbb{E} \left( \chi \left( \bigcap_{p=1}^{d-k} H(X_p) \cap K \right) \chi \left( \bigcap_{q=d-k-j+1}^{2d-k-l-j} H(X_q) \cap K \right) \right). \end{aligned}$$

Here, we have used that the summand for  $j=0$  equals  $\mathbb{E} \Psi_k^{(d)}(K_\varrho) \mathbb{E} \Psi_l^{(d)}(K_\varrho)$ . Therefore, the covariance  $\text{Cov}(\Psi_k^{(d)}(K_\varrho), \Psi_l^{(d)}(K_\varrho))$  can be written as non-negative polynomial of degree  $2d-k-l-1$  in  $\varrho \geq 0$ . Hence, in view of (3.1), dividing this polynomial by  $\varrho^{2d-k-l-1}$  yields the covariance  $\text{Cov}(\bar{\Psi}_{k,\varrho}^{(d)}(K), \bar{\Psi}_{l,\varrho}^{(d)}(K))$  which together with

$$\mathbb{E} \left( \chi \left( \bigcap_{p=1}^{d-k} H(X_p) \cap K \right) \chi \left( \bigcap_{q=d-k}^{2d-k-l-1} H(X_q) \cap K \right) \right) = \mathbb{E} g_{\chi, \Theta}^{(k,d)}(X_0, K) g_{\chi, \Theta}^{(l,d)}(X_0, K)$$

implies the limiting relation (3.3).

Since  $\chi \left( \bigcap_{q=d-k}^{2d-k-l-1} H(X_q) \cap K \right) \geq \chi \left( \bigcap_{q=d-k}^{2d-2k-1} H(X_q) \cap K \right)$  for  $k \leq l$ , we get

$$\mathbb{E} g_{\chi, \Theta}^{(k,d)}(X_0, K) g_{\chi, \Theta}^{(l,d)}(X_0, K) \geq \mathbb{E} (g_{\chi, \Theta}^{(k,d)}(X_0, K))^2 \geq (\mathbb{E} g_{\chi, \Theta}^{(k,d)}(X_0, K))^2$$

and the non-degeneracy of  $\Theta(\cdot)$  entails that  $\mathbb{E} \Psi_k^{(d)}(K) > 0$  for any convex body  $K$  with inner points, see Chapt. 6.3 in [23], so that, by (2.7),  $\mathbb{E} (g_{\chi, \Theta}^{(k,d)}(X_0, K))^2 > 0$  for  $k = 0, 1, \dots, d-1$ .

Finally, relation (3.4) follows by combining (3.2) and (3.3), where the distribution of  $X_0$  is  $Q_U^{(1)}(\cdot)$  as defined in Sect. 2. Thus, Lemma 3.1 is proved.  $\square$

We now establish the announced multivariate CLT for the random counting variables (2.1). For this we consider the  $d$ -dimensional vector of centered and individually scaled counting variables (3.1). By  $\mathcal{N}_d(o, \Sigma)$  we denote a  $d$ -dimensional Gaussian vector with mean vector  $o = (0, \dots, 0)^\top$  and covariance matrix  $\Sigma$  and  $\xrightarrow{d}$  means convergence in distribution.

**Theorem 3.1** *Let  $\Psi_{\lambda, \Theta}^{(d)}$  be a stationary PHP with non-degenerate  $\Theta(\cdot)$ . Then,*

$$(\overline{\Psi}_{k, \varrho}^{(d)}(K))_{k=0}^{d-1} \xrightarrow[\varrho \rightarrow \infty]{d} \mathcal{N}_d(o, \Sigma_d(\Theta, K)) , \quad (3.5)$$

where the entries of the covariance matrix  $\Sigma_d(\Theta, K) = (\sigma_{kl}^{(d)}(\Theta, K))_{k, l=0}^{d-1}$  are given by the limits (3.3). If  $\Theta(\cdot)$  is the uniform distribution on  $\mathbf{S}_+^{d-1}$ , then

$$(\overline{\Psi}_{k, \varrho}^{(d)}(K))_{k=0}^{d-1} \xrightarrow[\varrho \rightarrow \infty]{d} \mathcal{N}_d(o, \Sigma_d(K)) , \quad (3.6)$$

where  $\mu_{\Theta}^{(k, d)}(K) = \mu_U^{(k, d)}(K)$  is as in (2.8) and  $\Sigma_d(K) = (\sigma_{kl}^{(d)}(K))_{k, l=0}^{d-1}$  is defined by (3.4).

**Proof** For notational ease put  $N_{\varrho} = N(K_{\varrho})$ ,  $n_{\varrho} = \mathbb{E}N(K_{\varrho}) = \lambda \varrho b_{\Theta}(K)$  and  $\mu_{d-k}(f) = \mathbb{E}f(X_1, \dots, X_{d-k})$ . We first apply the decomposition (2.11) for general  $U$ -statistics to the counting variables (2.1): For  $k = 0, 1, \dots, d-1$ ,

$$\begin{aligned} \Psi_k^{(d)}(K_{\varrho}) - \mathbb{E}\Psi_k^{(d)}(K_{\varrho}) &\stackrel{d}{=} \left( \binom{N_{\varrho}}{d-k} - \frac{n_{\varrho}^{d-k}}{(d-k)!} \right) \mu_{d-k}(f) \\ &+ \binom{N_{\varrho}-1}{d-k-1} \sum_{i=1}^{N_{\varrho}} \left( g_{\chi, \Theta}^{(k, d)}(X_i, K) - \mu_{d-k}(f) \right) + \binom{N_{\varrho}}{d-k} R_{N_{\varrho}}^{(d-k)}(f) , \end{aligned}$$

where  $f(X_1, \dots, X_{d-k}) = \chi(H(X_1) \cap \dots \cap H(X_{d-k}) \cap K)$  and the  $X_1, X_2, \dots$  are IID random vectors in  $\mathbb{R}^1 \times \mathbf{S}_+^{d-1}$  with common distribution  $Q_{\Theta}^{(1)}(\cdot)$ .

Dividing both sides of the previous equality by  $\varrho^{d-k-1/2}$  and some simple rearrangements on the left-hand side provide

$$\begin{aligned}
\overline{\Psi}_{k,\varrho}^{(d)}(K) &\stackrel{d}{=} \frac{1}{\varrho^{d-k-1}} \binom{N_\varrho - 1}{d-k-1} \frac{1}{\sqrt{\varrho}} \left( \sum_{i=1}^{N_\varrho} g_{\chi,\Theta}^{(k,d)}(X_i, K) - n_\varrho \mu_{d-k}(f) \right) \\
&+ \frac{\mu_{d-k}(f)}{\varrho^{d-k-1/2}} \left( \binom{N_\varrho}{d-k} - N_\varrho \binom{N_\varrho - 1}{d-k-1} + n_\varrho \binom{N_\varrho - 1}{d-k-1} - \frac{n_\varrho^{d-k}}{(d-k)!} \right) \\
&+ \frac{1}{\varrho^{d-k-1/2}} \binom{N_\varrho}{d-k} R_{N_\varrho}^{(d-k)}(f) .
\end{aligned}$$

Since  $N_\varrho$  is conditionally independent of  $X_1, X_2, \dots$ , we find, using Lemma 2.1, that

$$\mathbb{E} \left( (R_{N_\varrho}^{(d-k)}(f))^2 \mid N_\varrho = n \right) \leq \frac{c_{d-k}}{n^2} \mathbb{E} f^2(X_1, \dots, X_{d-k}) \quad \text{for } n \geq d-k .$$

Hence, we deduce that

$$\begin{aligned}
\mathbb{E} \left( \binom{N_\varrho}{d-k} R_{N_\varrho}^{(d-k)}(f) \right)^2 &= \sum_{n \geq d-k} \binom{n}{d-k}^2 \mathbb{E} \left( (R_{N_\varrho}^{(d-k)}(f))^2 \mid N_\varrho = n \right) \mathbb{P}(N_\varrho = n) \\
&\leq \frac{c_{d-k} \mathbb{E} f^2(X_1, \dots, X_{d-k})}{(d-k)^2} \mathbb{E} \binom{N_\varrho - 1}{d-k-1}^2 .
\end{aligned}$$

Since  $\mathbb{E}((N_\varrho - 1)(N_\varrho - 2) \cdots (N_\varrho - d + k + 1))^2$  is a polynomial of degree  $2d - 2k - 2$  in  $n_\varrho$ , it follows that

$$\frac{1}{\varrho^{2d-2k-1}} \mathbb{E} \binom{N_\varrho - 1}{d-k-1}^2 \xrightarrow{\varrho \rightarrow \infty} 0 .$$

Thus,

$$\frac{1}{\varrho^{d-k-1/2}} \binom{N_\varrho}{d-k} R_{N_\varrho}^{(d-k)}(f) \xrightarrow[\varrho \rightarrow \infty]{\mathbb{P}} 0 , \quad (3.7)$$

where  $\xrightarrow{\mathbb{P}}$  denotes convergence in probability  $\mathbb{P}$ . The validity of the limit

$$\frac{1}{\varrho^{d-k-1/2}} \left( \binom{N_\varrho}{d-k} - N_\varrho \binom{N_\varrho - 1}{d-k-1} + n_\varrho \binom{N_\varrho - 1}{d-k-1} - \frac{n_\varrho^{d-k}}{(d-k)!} \right) \xrightarrow[\varrho \rightarrow \infty]{\mathbb{P}} 0 \quad (3.8)$$

can be verified by repeating word by word the arguments used in Sect. 3 of [9] to treat the special case  $K = B_1^d$ . With

$$G_{k,\varrho}^{(d)}(K) = \varrho^{-1/2} \left( \sum_{i=1}^{N_\varrho} g_{\chi,\Theta}^{(k,d)}(X_i, K) - n_\varrho \mu_{d-k}(f) \right) \quad \text{for } k = 0, 1, \dots, d-1$$

and by using that  $N_\varrho$  is Poisson distributed with mean  $\lambda \varrho b_\Theta(K)$  and independent of  $X_1, X_2, \dots$ , it is easily checked that

$$\mathbb{E} G_{k,\varrho}^{(d)}(K) G_{l,\varrho}^{(d)}(K) = \lambda b_{\Theta}(K) \mathbb{E} g_{\chi,\Theta}^{(k,d)}(X_0, K) g_{\chi,\Theta}^{(l,d)}(X_0, K) \quad (3.9)$$

for  $0 \leq k \leq l \leq d-1$  as well as  $N_{\varrho}/\varrho \xrightarrow[\varrho \rightarrow \infty]{\text{P}} \lambda b_{\Theta}(K)$ , which in turn implies the limit

$$\frac{1}{\varrho^{d-k-1}} \binom{N_{\varrho}-1}{d-k-1} \xrightarrow[\varrho \rightarrow \infty]{\text{P}} \frac{(\lambda b_{\Theta}(K))^{d-k-1}}{(d-k-1)!}.$$

Combining (3.7), (3.8), (3.9) for  $k = l$ , and the latter relation, we obtain with the above-introduced notation

$$\overline{\Psi}_{k,\varrho}^{(d)}(K) \stackrel{\text{d}}{=} \frac{(\lambda b_{\Theta}(K))^{d-k-1}}{(d-k-1)!} G_{k,\varrho}^{(d)}(K) + Z_{k,\varrho}^{(d)}(K),$$

where the term  $Z_{k,\varrho}^{(d)}(K)$  disappears asymptotically as  $\varrho \rightarrow \infty$ , i.e.  $Z_{k,\varrho}^{(d)}(K) \xrightarrow[\varrho \rightarrow \infty]{\text{P}} 0$ .

Recall that due to well-known Cramér–Wold device, see [2], the multivariate CLT (3.5) is equivalent to the univariate CLT

$$\sum_{k=0}^{d-1} t_k \overline{\Psi}_{k,\varrho}^{(d)}(K) \xrightarrow[\varrho \rightarrow \infty]{\text{d}} \mathcal{N}(0, t^{\top} \Sigma_d(\Theta, K) t) \quad (3.10)$$

for all  $t = (t_0, \dots, t_{d-1})^{\top} \in \mathbb{R}^d \setminus \{o\}$ . Slutsky's theorem, see also [2], tells us that on the left-hand side of (3.10) the random variables  $\overline{\Psi}_{k,\varrho}^{(d)}(K)$  can be replaced by the scaled random sums  $(\lambda b_{\Theta}(K))^{d-k-1} G_{\varrho}^{(k,d)}(K)/(d-k-1)!$  for  $k = 0, 1, \dots, d-1$  without changing the limit in distribution. In other words, we have to prove

$$H_{\varrho}^{(d)} = \varrho^{-1/2} \left( \sum_{i=1}^{N_{\varrho}} h(X_i) - n_{\varrho} \mathbb{E} h(X_0) \right) \xrightarrow[\varrho \rightarrow \infty]{\text{d}} \mathcal{N}(0, t^{\top} \Sigma(\Theta, K) t) \quad (3.11)$$

for all  $t = (t_0, \dots, t_{d-1})^{\top} \in \mathbb{R}^d \setminus \{o\}$ , where

$$h(X_i) = \sum_{k=0}^{d-1} t_k \frac{(\lambda b_{\Theta}(K))^{d-k-1}}{(d-k-1)!} g_{\chi,\Theta}^{(k,d)}(X_i, K) \quad \text{for } i = 1, 2, \dots.$$

This means that the proof of (3.5) can be put down to a CLT for sums of a Poisson distributed number of IID random variables. There are several results in the literature addressing this problem in great generality. However, we can do this in a simple way without referring to other results. Note that after a short calculation, using among others that  $\mathbb{E} N_{\varrho}(N_{\varrho}-1) = n_{\varrho}^2$ , we arrive at

$$\mathbb{E}(H_{\varrho}^{(d)})^2 = \lambda b_{\Theta}(K) \mathbb{E} h^2(X_0) = \sum_{k=0}^{d-1} \sum_{l=0}^{d-1} t_k t_l \sigma_{kl}^{(d)} = t^{\top} \Sigma(\Theta, K) t$$

in accordance with the expressions of the asymptotic covariances in (3.3).

The characteristic function of  $H_\varrho^{(d)}$  is easily obtained by using the independence assumptions and generating function  $\mathbb{E} z^{N_e} = \exp\{n_\varrho(z-1)\}$  for any complex  $z$ , so that

$$\mathbb{E} \exp\left\{is H_\varrho^{(d)}\right\} = \exp\left\{n_\varrho \left( \mathbb{E} \exp\left\{\frac{is}{\sqrt{\varrho}} h(X_0)\right\} - 1 - \frac{is}{\sqrt{\varrho}} \mathbb{E} h(X_0) \right)\right\}$$

whence, by applying the well-known inequality  $|e^{ix} - 1 - ix - \frac{(ix)^2}{2}| \leq \frac{|x|^3}{6}$  for  $x \in \mathbb{R}^1$ , it follows that, for all  $s \in \mathbb{R}^1$ ,

$$\mathbb{E} \exp\left\{is H_\varrho^{(d)}\right\} \xrightarrow{\varrho \rightarrow \infty} \exp\left\{-\frac{s^2}{2} \lambda_{b_\Theta(K)} \mathbb{E} h^2(X_0)\right\} = \exp\left\{-\frac{s^2}{2} t^\top \Sigma(\Theta, K) t\right\},$$

which is equivalent to (3.11). Hence, the first assertion of Theorem 3.1 is proved. The second assertion is an immediate consequence of the first one and (3.4) which completes the proof of Theorem 3.1.  $\square$

## 4 CLTs for the total $k$ -volume of intersection $k$ -flats in $\varrho K$

In continuation of the previous section we now consider the joint asymptotic behaviour (as  $\varrho \rightarrow \infty$ ) of the centered and scaled random variables

$$\bar{\zeta}_{k,\varrho}^{(d)}(K) = \varrho^{-(d-1/2)} \left( \zeta_k^{(d)}(K_\varrho) - \lambda^{d-k} \lambda_\Theta^{(k,d)} \varrho^d \nu_d(K) \right) \quad (4.1)$$

for  $k = 0, 1, \dots, d-1$  for a stationary PHP  $\Psi_{\lambda,\Theta}^{(d)}$ , where  $\lambda_\Theta^{(k,d)}$  defined by (2.9) resp. by (2.10) if the PHP is additionally isotropic. In order to argue along the line of the previous section we need the conditional expectation

$$g_{\nu,\Theta}^{(k,d)}((p,v), K) = \mathbb{E} \nu_k \left( \bigcap_{i=1}^{d-k-1} H(X_i) \cap H(p,v) \cap K \right)$$

for  $(p,v) \in \mathbb{R}^1 \times \mathbf{S}_+^{d-1}$ . In order to evaluate  $g_{\nu,U}^{(k,d)}((p,v), K)$  by analogy to (3.2), we apply Crofton's formula (2.6)  $d-k-1$  times (with  $H(p,v) \cap K$  instead of  $K$ ) leading to

$$g_{\nu,U}^{(k,d)}((p,v), K) = \left( \frac{\kappa_{d-1}}{d \kappa_d} \right)^{d-k} \frac{d! \kappa_d}{k! \kappa_k (b(K))^{d-k-1}} V_{d-1}^{(d)}(H(p,v) \cap K). \quad (4.2)$$

for  $k = 0, 1, \dots, d-1$ . As in the foregoing section we first calculate the asymptotic covariances

$$\tau_{kl}^{(d)}(\Theta, K) = \lim_{\varrho \rightarrow \infty} \text{Cov}(\bar{\zeta}_{k,\varrho}^{(d)}(K), \bar{\zeta}_{l,\varrho}^{(d)}(K)).$$

For brevity, put  $\tau_{kl}^{(d)}(K) = \tau_{kl}^{(d)}(U, K)$  for  $k, l = 0, 1, \dots, d-1$ .



**Lemma 4.1** *Let  $\Psi_{\lambda, \Theta}^{(d)}$  be a stationary PHP with non-degenerate  $\Theta(\cdot)$ . Then,*

$$\tau_{kl}^{(d)}(\Theta, K) = \frac{(\lambda b_{\Theta}(K))^{2d-k-l-1}}{(d-k-1)!(d-l-1)!} \mathbb{E} g_{\nu, \Theta}^{(k,d)}(X_0, K) g_{\nu, \Theta}^{(l,d)}(X_0, K) \quad (4.3)$$

for  $k, l = 0, 1, \dots, d-1$ .

In case of a motion-invariant PHP  $\Psi_{\lambda, U}^{(d)}$ , we get with  $\lambda_U^{(k,d)}$  from (2.10) that

$$\begin{aligned} \tau_{kl}^{(d)}(K) &= \lambda^{2d-k-l-1} \lambda_U^{(k,d)} \lambda_U^{(l,d)} (d-k)(d-l) \\ &\times \int_{\mathbf{S}_+^{d-1}} \int_{\mathbf{R}^1} (V_{d-1}^{(d)}(H(p, v) \cap K))^2 dp U(dv). \end{aligned} \quad (4.4)$$

**Proof** The proof of Lemma 4.1 resembles that of Lemma 3.1 almost verbatim. We have to replace the kernel function  $\chi(H(X_1^{(\varrho)} \cap \dots \cap H(X_1^{(\varrho)} \cap K_{\varrho}))$  by  $\nu_k(H(X_1^{(\varrho)} \cap \dots \cap H(X_1^{(\varrho)} \cap K_{\varrho}))$  and then to take into consideration the different scaling

$$\begin{aligned} &\mathbb{E} \nu_k \left( \bigcap_{p=1}^{d-k} H(X_{i_p}^{(\varrho)}) \cap K_{\varrho} \right) \nu_l \left( \bigcap_{q=d-k-j+1}^{2d-k-l-j} H(X_{i_q}^{(\varrho)}) \cap K_{\varrho} \right) \\ &= \varrho^{k+l} \mathbb{E} \left( \nu_k \left( \bigcap_{p=1}^{d-k} H(X_p) \cap K \right) \nu_l \left( \bigcap_{q=d-k-j+1}^{2d-k-l-j} H(X_q) \cap K \right) \right). \end{aligned}$$

Therefore, the covariance  $\text{Cov}(\zeta_k^{(d)}(K_{\varrho}) \zeta_l^{(d)}(K_{\varrho}))$  represents a polynomial of degree  $2d-1$  in  $\varrho$ , so that after dividing by  $\varrho^{2d-1}$  we get the limit (4.3) whence, by (4.2), it follows (4.4) completing the proof of Lemma 4.1.  $\square$

**Theorem 4.1** *Let  $\Psi_{\lambda, \Theta}^{(d)}$  be a stationary PHP with non-degenerate  $\Theta(\cdot)$ . Then,*

$$(\bar{\zeta}_{k, \varrho}^{(d)}(K))_{k=0}^{d-1} \xrightarrow[\varrho \rightarrow \infty]{d} \mathcal{N}_d(o, T_d(\Theta, K)), \quad (4.5)$$

where the entries of the covariance matrix  $T_d(\Theta, K) = (\tau_{kl}^{(d)}(\Theta, K))_{k, l=0}^{d-1}$  are given by the limits (4.3). If  $\Theta(\cdot)$  is the uniform distribution on  $\mathbf{S}_+^{d-1}$ , then

$$(\bar{\zeta}_{k, \varrho}^{(d)}(K))_{k=0}^{d-1} \xrightarrow[\varrho \rightarrow \infty]{d} \mathcal{N}_d(o, T_d(K)), \quad (4.6)$$

where  $\lambda_{\Theta}^{(k,d)} = \lambda_U^{(k,d)}$  is taken from (2.10) and  $T_d(K) = (\tau_{kl}^{(d)}(K))_{k, l=0}^{d-1}$  is determined by (4.4).

Employing Lemma 4.1, the proof of the CLT (4.5) (and also of (4.6)) coincides step by step - up to evident changes - with the proof of the first assertion of Theorem 3.1. For this reason the details of the proof of Theorem 4.1 are left to the reader. Instead we make some remarks on a more explicit representation of the conditional expectation  $g_{\nu, \Theta}^{(k,d)}((p, v), K)$  the derivation of which can be found in [11].

**Remark** Let  $\Psi_{\lambda, \Theta}^{(d)}$  be a stationary, non-degenerate PHP in  $\mathbb{R}^d$ . Then

$$g_{\nu, \Theta}^{(k,d)}((p, v), K) = \frac{(d-k-1)!}{(b_{\Theta}(K))^{d-k-1}} V_{d-k-1}^{(d-1)}(Z_{\Theta}^v) V_{d-1}^{(d)}(H(p, v) \cap K)$$

for  $k = 0, 1, \dots, d-1$  and all  $(p, v) \in \mathbb{R}^1 \times \mathbf{S}_+^{d-1}$  such that  $H(p, v) \cap K \neq \emptyset$ , where  $Z_{\Theta}^v$  denotes the image of the associated zonoid  $Z_{\Theta}$  under orthogonal projection onto  $H(0, v)$ . With the notation introduced in at the end of Section 2, the intrinsic volume  $V_{d-k-1}^{(d-1)}(Z_{\Theta}^v)$  is seen to be equal to

$$\frac{1}{(d-k-1)!} \int_{\mathbf{S}_+^{d-1}} \cdots \int_{\mathbf{S}_+^{d-1}} \nabla_{d-k}(v_1, \dots, v_{d-k-1}, v) \Theta(dv_1) \cdots \Theta(dv_{d-k-1}).$$

Consequently, the covariance  $\tau_{kl}^{(d)}(\Theta, K)$  in (4.3) admits the representation

$$\lambda^{2d-k-l-1} \int_{\mathbf{S}_+^{d-1}} \int_{\mathbb{R}^1} (V_{d-1}^{(d)}(H(p, v) \cap K))^2 dp V_{d-k-1}^{(d-1)}(Z_{\Theta}^v) V_{d-l-1}^{(d-1)}(Z_{\Theta}^v) \Theta(dv)$$

for  $k, l = 0, 1, \dots, d-1$ . In view of  $V_{d-1}^{(d)}(H(p, v) \cap K) = \nu_{d-1}(H(p, v) \cap K)$ , this formula as well as the CLT (4.5) hold for any bounded  $K \in \mathfrak{B}^d$  satisfying  $\nu_d(K) > 0$ .

## 5 Some properties of the matrices $\Sigma_d(K)$ and $T_d(K)$

In this section we study the algebraic properties of the covariance matrices  $\Sigma_d(K)$  and  $T_d(K)$  defined in Theorem 3.1 and 4.1, respectively. Whereas the linear hull spanned by the columns of  $T_d(K)$  is a one-dimensional subspace in  $\mathbb{R}^d$  for any  $d \geq 2$ , the rank of  $\Sigma_d(K)$  equals  $d$ , i.e.  $\det(\Sigma_d(K)) > 0$ .

**Theorem 5.1** *Let  $\Psi_{\lambda, U}^{(d)}$  be a motion-invariant PHP in  $\mathbb{R}^d$  with intensity  $\lambda > 0$ , and let  $K$  be a convex body in  $\mathbb{R}^d$  containing  $B_{\varepsilon}^d$  for some  $\varepsilon > 0$ . Then,*

- (i) *the rank of  $T_d(K)$  equals 1 for any  $d \geq 2$  and, for  $0 \leq k < l \leq d-1$ ,*

$$\frac{\bar{\zeta}_{l, \varrho}^{(d)}(K)}{\sqrt{\tau_{ll}^{(d)}(K)}} - \frac{\bar{\zeta}_{k, \varrho}^{(d)}(K)}{\sqrt{\tau_{kk}^{(d)}(K)}} \xrightarrow[\varrho \rightarrow \infty]{\mathbb{P}} 0, \quad (5.1)$$

- (ii) *and  $\Sigma_d(K)$  has full rank  $d$ , i.e. the inverse  $(\Sigma_d(K))^{-1}$  exists.*

**Proof** Clearly, (4.6) implies that  $\tau_{kl}^{(d)}(K) = \sqrt{\tau_{kk}^{(d)}(K) \tau_{ll}^{(d)}(K)}$  for  $0 \leq k, l \leq d-1$ . This means that each column of the matrix  $T_d(K)$  is a multiple of the column vector  $(\sqrt{\tau_{kk}^{(d)}(K)})_{k=0}^{d-1}$  proving (i). The relation (5.1) even holds in the quadratic mean since the shape of the limiting covariances (4.6) implies that

$$\mathbb{E} \left( \sqrt{\tau_{kk}^{(d)}(K)} \bar{\zeta}_{l,\varrho}^{(d)}(K) - \sqrt{\tau_{ll}^{(d)}(K)} \bar{\zeta}_{k,\varrho}^{(d)}(K) \right)^2 \xrightarrow{\varrho \rightarrow \infty} 0.$$

Going back to the very definition of positive definiteness und making use of (3.4), our assertion (ii) means that

$$\sum_{k,l=0}^{d-1} t_k t_l \sigma_{kl}^{(d)}(K) = \mathbb{E} \left( \sum_{k=0}^{d-1} s_k V_{d-k-1}^{(d)}(H(X_0) \cap K) \right)^2 > 0 \quad (5.2)$$

for any  $(t_0, t_1, \dots, t_{d-1}) \in \mathbb{R}^d \setminus \{0\}$  with  $s_k = \lambda^{d-k-1} \sqrt{\lambda b(K)} a_k^{(d)} t_k$  for  $k = 0, 1, \dots, d-1$ . Equivalently, the (non-negative) double sum in the foregoing line attains zero iff  $t_0 = \dots = t_{d-1} = 0$ . Assuming that the right-hand side of (5.2) disappears, i.e.,

$$\sum_{k=0}^{d-1} s_k V_{d-k-1}^{(d)}(H(p, v) \cap K) = 0 \quad (5.3)$$

for  $(\nu_1 \times U)$ -almost every  $(p, v) \in \mathbb{R}^1 \times \mathbf{S}_+^{d-1}$  satisfying  $H(p, v) \cap K \neq \emptyset$ , we will prove that  $s_0 = \dots = s_{d-1} = 0$ .

Let  $e \in \partial K$  be an *extreme point* of  $K$ . Applying the characterization of extreme points given in [21], Lemma 1.4.6, we find, for any  $\delta > 0$ , a closed halfspace  $H_{q,u}^+ = \{H(p, u) : p \geq q\}$  with  $q = q(e, \delta) \in \mathbb{R}^1$  and  $u = u(e, \delta) \in \mathbf{S}_+^{d-1}$  depending on  $(e, \delta)$ , such that

$$d(e, \delta) := \inf\{\|e - x\| : x \in H(q, u)\} > 0 \quad \text{and} \quad K \cap H_{q,u}^+ \subset B_\delta^d(e) := B_\delta^d + e.$$

Since  $B_\varepsilon^d \subseteq K$ , that part of the convex hull of  $\{e\}$  and  $B_\varepsilon^d$  contained in the halfspace  $H_{q,u}^+$  also belongs to  $B_\delta^d(e)$ . Therefore, by evident continuity arguments (shifting  $H(q, u)$  closer to  $e$  and moving  $q$  slightly) we find an open interval  $(\eta_1, \eta_2) \subset [q, q + d(e, \delta)]$  and a sufficiently small  $\eta$ -neighbourhood  $W_\eta(q) := B_\eta^d(u) \cap \mathbf{S}_+^{d-1}$  of  $u \in \mathbf{S}_+^{d-1}$  such that  $V_{d-1}^{(d)}(H(p, v) \cap K) > 0$  and  $V_1^{(d)}(H(p, v) \cap K) \leq V_1^{(d)}(B_\delta^d) = \delta d \kappa_d / \kappa_{d-1}$  for all  $(p, v) \in (\eta_1, \eta_2) \times W_\eta(q)$ . Note that the latter set has positive  $(\nu_1 \times U)$ -measure and  $\delta > 0$  can be chosen arbitrarily small. Now we use a well-known consequence of the Alexandrov-Fenchel inequality, see [21], Chapt. 6.4, which can be rewritten in terms of the intrinsic volumes  $V_k^{(d)}(\cdot)$  as follows:

$$V_k^{(d)}(\cdot) \leq \left( V_j^{(d)}(\cdot) \right)^{k/j} \binom{d}{k} \frac{\kappa_d}{\kappa_{d-k}} \left( \binom{d}{j} \frac{\kappa_d}{\kappa_{d-j}} \right)^{-k/j} \quad (5.4)$$

for  $0 \leq j \leq k \leq d-1$ .

We first show that  $s_{d-1} = 0$  in (5.3). Together with  $V_0^{(d)}(H(p, v) \cap K) = 1$  it follows from (5.3) that

$$|s_{d-1}| \leq \sum_{k=1}^{d-1} |s_{d-k-1}| V_k^{(d)}(H(p, v) \cap K). \quad (5.5)$$

Inequality (5.4) for  $j = 1$  applied to  $H(p, v) \cap K$  for  $(p, v) \in (\eta_1, \eta_2) \times W_\eta(q)$  yields

$$V_k^{(d)}(H(p, v) \cap K) \leq \frac{\delta^k \kappa_d}{\kappa_{d-k}} \binom{d}{k} \quad \text{for } 1 \leq k \leq d-1. \quad (5.6)$$

Inserting these estimates on the right-hand side of (5.5) and letting  $\delta \downarrow 0$  reveal that only  $s_{d-1} = 0$  is possible. In the next step, we assume that  $s_{d-1} = \dots = s_{d-j} = 0$  for some  $j \in \{1, \dots, d-2\}$  so that then (5.3) implies

$$|s_{d-j-1}| \leq \sum_{k=j+1}^{d-1} |s_{d-k-1}| \frac{V_k^{(d)}(H(p, v) \cap K)}{V_j^{(d)}(H(p, v) \cap K)}. \quad (5.7)$$

Combining (5.4) with (5.5) (the latter for  $k = j$ ) leads to the estimate

$$\frac{V_k^{(d)}(H(p, v) \cap K)}{V_j^{(d)}(H(p, v) \cap K)} \leq \delta^{k-j} \frac{j! (d-j)! \kappa_{d-j}}{k! (d-k)! \kappa_{d-k}} \quad \text{for } (p, v) \in (\eta_1, \eta_2) \times W_\eta(q),$$

which together with (5.7) enables us to conclude that  $s_{d-j-1} = 0$ . This proves (ii) and completes the proof of Theorem 5.1.  $\square$

## 6 Special cases and some inequalities

In this final section we derive some estimates and discuss extremal properties of the ovoid functionals

$$J_k^{(d)}(K) = \int_{\mathbb{S}_+^{d-1}} \int_{\mathbb{R}^1} (V_k^{(d)}(H(p, v) \cap K))^2 dp U(dv), \quad k = 0, 1, \dots, d-1,$$

which are motion-invariant, non-decreasing and continuous, but not additive on the family of convex bodies in  $\mathbb{R}^d$ .

We first compute  $J_k^{(d)}(K)$  for some special convex bodies such as balls  $B_r^d$  in any dimension, ellipses  $E_{a,b} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2/a^2 + x_2^2/b^2 \leq 1\}$  with numerical excentricity  $k = \sqrt{1 - b^2/a^2} \in [0, 1)$  (i.e.  $a \geq b$ ) and planar rectangles  $R_{a,b} = [-a, a] \times [-b, b]$ .

### Lemma 6.1

$$J_k^{(d)}(B_r^d) = \left( \frac{(d-1)! \kappa_{d-1}}{(d-k-1)! \kappa_{d-k-1}} \right)^2 \frac{(2r)^{2k+1}}{(2k+1)!}, \quad 0 \leq k \leq d-1. \quad (6.1)$$

$$J_1^{(2)}(E_{a,b}) = \frac{32ab^2}{3\pi} F\left(\frac{\pi}{2}, k\right), \quad \text{where } F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad (6.2)$$

is a ‘complete elliptic integral of the first kind in Legendre form’.

$$J_1^{(2)}(R_{a,b}) = \frac{16}{3\pi} \left( I(a,b) + I(b,a) \right), \quad (6.3)$$

where

$$I(a,b) = 3ab^2 \log\left(\frac{\sqrt{a^2+b^2}+a}{b}\right) - b^2(\sqrt{a^2+b^2}-b).$$

The formulae (6.1) were derived in [9]. To calculate the functionals  $J_1^{(2)}(E_{a,b})$  and  $J_1^{(2)}(R_{a,b})$  (which are needed in cases as illustrated in Figure 1) one has to determine the lengths of all chords through  $E_{a,b}$  and  $R_{a,b}$ , respectively. In particular, the computation of the chord lengths through  $R_{a,b}$  requires to distinguish several cases of how a (random) line crosses the boundary  $\partial R_{a,b}$ . These lengthy procedures and the computations of the corresponding integrals might be of interest in its own right and will be carried out in the Appendix below.

In the particular case  $a = b$  relation (6.2) coincides with (6.1) for  $d = 2, k = 1, r = a$  and (6.3) results in

$$J_1^{(2)}(R_{a,a}) = \frac{32a^3}{3\pi} \left( 3\log(1+\sqrt{2}) + 1 - \sqrt{2} \right) \approx 7.5712a^3.$$

Notice that (6.2) and (6.3) exhibit special cases of *chord-power integrals* which are studied quite well in integral geometry, see [22] and references therein. From Theorem 6.3.7 in [22] we deduce the inequality

$$J_1^{(2)}(K) \leq \frac{16}{3} \left( \frac{\nu_2(K)}{\pi} \right)^{3/2} \quad (6.4)$$

for any convex body in  $\mathbb{R}^2$ , where equality holds iff  $K = B_r^2$ , see also [20] for more general inequalities. Having in mind the variances (3.4) and (4.4) for  $k = l = 1, d = 2$ , it is intuitively clear that, for fixed area  $\nu_2(K)$ , the functional  $J_1^{(2)}(K)$  may take arbitrarily small values when  $K$  becomes strip-like. For example, fixing the area  $A = \pi ab$  of the family of ellipses  $E_{a,b}$  and setting  $b = A/\pi a$ , we get

$$J_1^{(2)}(E_{a,b}) = \frac{32A^2}{3\pi^3 a} F\left(\frac{\pi}{2}, \sqrt{1 - A^2/\pi^2 a^4}\right) \xrightarrow{a \rightarrow \infty} 0.$$

This means that long and thin planar convex bodies  $K$  diminish the variance of  $\overline{\Psi}_{0,\varrho}^{(d)}(K)$ . Likewise, for fixed  $\nu_d(K)$ , the functional  $J_{d-1}^{(d)}(K)$  can be minimized to zero in each dimension  $d \geq 2$ . In contrast to this, the functional  $J_1^{(3)}(K)$  is bounded from below by  $\pi \nu_3(K)$ . This is rapidly seen by applying the isoperimetric inequality  $(V_1^{(2)}(H(p,v) \cap K))^2 \geq \pi V_2^{(2)}(H(p,v) \cap K)$ , see [21], p. 323, for all planar convex bodies  $H(p,v) \cap K$ . Together with Crofton’s formula (2.6) for  $d = 3, k = 2$  we obtain

$$J_1^{(3)}(K) \geq \pi \int_{\mathbf{S}_+^{d-1}} \int_{\mathbf{R}^1} V_2^{(3)}(H(p, v) \cap K) dp U(dv) = \pi \nu_3(K) ,$$

where the equality holds iff  $K = B_r^3$ .

So far, to the best of the author's knowledge, sharp upper bounds of  $J_k^{(d)}(K)$  ( $1 \leq k \leq d-1$ ,  $d \geq 3$ ) in terms of intrinsic volumes of  $K$  seem to be unknown. We conclude this paper with a conjecture concerning an upper bound of  $J_{d-1}^{(d)}(K)$  for any  $d \geq 2$ . The relations (6.1) and (6.4) give rise to presume that the estimate

$$J_{d-1}^{(d)}(K) \leq \frac{((d-1)! \kappa_{d-1})^2}{(2d-1)!} \left( \frac{2^d \nu_d(K)}{\kappa_d} \right)^{(2d-1)/d}$$

holds with equality for balls  $K = B_r^d$ .

## 7 Appendix: Proof of Lemma 6.1

### 1. Computation of $J_1^{(2)}(E_{a,b})$ for $a \geq b$

Unoriented straight lines  $g(p, \varphi)$  in a Cartesian  $xy$ -coordinate system can be represented in the form

$$g(p, \varphi) = \{ (x, y) \in \mathbf{R}^2 : x \cos \varphi + y \sin \varphi = p \} \quad \text{with} \quad 0 \leq \varphi \leq \pi, \quad p \in \mathbf{R}^1, \quad (7.1)$$

which corresponds to the representation of hyperplanes  $H(p, v)$  at the beginning of Section 2 with orientation vector  $v = (\cos \varphi, \sin \varphi) \in \mathbf{S}_+^1$ . For reasons of symmetry it suffices to consider chords with end-points  $g(p, \varphi) \cap \partial E_{a,b} = \{(x_1, y_1), (x_2, y_2)\}$  only for  $0 \leq \varphi \leq \pi/2$  and  $p \geq 0$ . The coordinates of the two points of intersection  $(x_1, y_1)$  and  $(x_2, y_2)$  are obtained as solutions of the system of equations

$$b^2 x^2 + a^2 y^2 = a^2 b^2 \quad \text{and} \quad x \cos \varphi + y \sin \varphi = p$$

for  $0 \leq \varphi < \pi/2$ ,  $0 \leq p \leq \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}$ .

By solving the corresponding quadratic equation we find that, for  $i = 1, 2$ ,

$$x_i = \frac{p - y_i \sin \varphi}{\cos \varphi}, \quad y_i = \frac{p b^2 \sin \varphi + (-1)^i a b \cos \varphi \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - p^2}}{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi},$$

whence it follows

$$y_2 - y_1 = \frac{2 a b \cos \varphi \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - p^2}}{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}$$

and

$$x_2 - x_1 = -\frac{2 a b \sin \varphi \sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - p^2}}{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}$$

so that the squared length of the chord connecting  $(x_1, y_1)$  and  $(x_2, y_2)$  can be expressed by

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = \frac{4a^2 b^2 (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - p^2)}{(a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^2}.$$

Finally, using the symmetry and integrating (over  $\varphi$ ) w.r.t. the uniform distribution on  $\mathbf{S}_+^1$ , we obtain

$$\begin{aligned} J_1^{(2)}(E_{a,b}) &= \frac{4}{\pi} \int_0^{\pi/2} \int_0^{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} ((x_2 - x_1)^2 + (y_2 - y_1)^2) dp d\varphi \\ &= \frac{16a^2 b^2}{\pi} \int_0^{\pi/2} \int_0^{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} \frac{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi - p^2}{(a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^2} dp d\varphi \\ &= \frac{32}{3\pi} a^2 b^2 \int_0^{\pi/2} \frac{d\varphi}{\sqrt{a^2 \cos^2 \varphi + b^2 \sin^2 \varphi}} \\ &= \frac{32}{3\pi} a b^2 \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} \quad , \quad \text{where} \quad k = \sqrt{1 - \frac{b^2}{a^2}}. \quad \square \end{aligned}$$

One can prove the following expansions and estimates of  $F\left(\frac{\pi}{2}, k\right)$  for any  $k \in [0, 1]$ :

$$F\left(\frac{\pi}{2}, k\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{k^{2n}}{4^{2n}} \binom{2n}{n}^2 \leq \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{k^{2n}}{4^n n!} \prod_{k=1}^n (4k - 3) = \frac{\pi}{2} (1 - k^2)^{-1/4}.$$

Here we have used the inequality  $\frac{n!}{4^n} \binom{2n}{n}^2 \leq (4n - 3) \cdot (4n - 7) \cdots 5 \cdot 1$  for any  $n \geq 1$ .

Inserting  $k^2 = 1 - b^2/a^2$  and taking into account that  $\nu_2(E_{a,b}) = \pi a b$  yield

$$J_1^{(2)}(E_{a,b}) \leq \frac{32}{3} \pi a b^2 \frac{\pi}{2} \left(\frac{a}{b}\right)^{1/2} = \frac{16}{3} (ab)^{3/2} = \frac{16}{3} \left(\frac{\nu_2(E_{a,b})}{\pi}\right)^{3/2},$$

which coincides with (6.4) for  $K = E_{a,b}$ .

## 2. Computation of $J_1^{(2)}(R_{a,b})$

By definition the edges of the rectangle  $R_{a,b}$  are parallel to the axis and its vertices in anti-clockwise ordering are  $(-a, -b)$ ,  $(a, -b)$ ,  $(a, b)$ , and  $(-a, b)$ . As in the case of ellipses we consider chords arising from the intersection of the boundary  $\partial R_{a,b}$  with the straight line  $g(p, \varphi)$  defined in (7.1) only for  $0 \leq \varphi \leq \pi/2$  and  $p \geq 0$ . By contrast with ellipses, several different types of chords through  $R_{a,b}$  emerge. Basically, we distinguish case (I)  $0 \leq \varphi < x(a, b) := \arctan \frac{a}{b}$  and case (II)  $x(a, b) \leq \varphi < \frac{\pi}{2}$ .

In case (I), for  $\varphi$  being fixed, there exist certain  $p_1$  and  $p^*$  depending on  $a, b, \varphi$  such that the end-points of the chord  $g(p, \varphi) \cap R_{a,b}$  are  $\{(x_1, -b), (x_2, b)\}$  resp.  $\{(a, y_1), (x_2, b)\}$  for  $0 \leq p \leq p_1$  resp.  $p_1 \leq p \leq p^*$ , where

$$x_1 \cos \varphi - b \sin \varphi = p, \quad x_2 \cos \varphi + b \sin \varphi = p, \quad a \cos \varphi + y_1 \sin \varphi = p$$

and  $p_1$  resp.  $p^*$  is just the distances of the line containing  $(a, -b)$  resp.  $(a, b)$  from the origin, that is,

$$p_1 = p_1(a, b, \varphi) = a \cos \varphi - b \sin \varphi \quad \text{and} \quad p^* = p^*(a, b, \varphi) = a \cos \varphi + b \sin \varphi.$$

Hence, for  $0 \leq p \leq p_1(a, b, \varphi)$ , we get

$$(x_2 - x_1)^2 + (2b)^2 = \left( \frac{2b \sin \varphi}{\cos \varphi} \right)^2 + (2b)^2 = 4b^2 (1 + \tan^2 \varphi) = \frac{4b^2}{\cos^2 \varphi}$$

and, for  $p_1(a, b, \varphi) \leq p \leq p^*(a, b, \varphi)$ ,

$$(x_2 - a)^2 + (b - y_1)^2 = \left( \frac{p - b \sin \varphi}{\cos \varphi} - a \right)^2 + \left( \frac{p - a \cos \varphi}{\sin \varphi} - b \right)^2 = \left( \frac{p - a \cos \varphi - b \sin \varphi}{\sin \varphi \cos \varphi} \right)^2.$$

Now, we are in a position to evaluate the integral

$$K(a, b) = \int_0^{x(a,b)} \int_0^{p^*(a,b,\varphi)} (\nu_1(g(p, \varphi) \cap R_{a,b}))^2 dp d\varphi.$$

The first step is to calculate the inner integral

$$\begin{aligned} \int_0^{p^*(a,b,\varphi)} (\nu_1(g(p, \varphi) \cap R_{a,b}))^2 dp &= \int_0^{p_1(a,b,\varphi)} \frac{4b^2}{\cos^2 \varphi} dp + \int_{p_1(a,b,\varphi)}^{p^*(a,b,\varphi)} \left( \frac{p^*(a,b,\varphi) - p}{\sin \varphi \cos \varphi} \right)^2 dp \\ &= \frac{4b^2 p_1(a,b,\varphi)}{\cos^2 \varphi} + \frac{(p^*(a,b,\varphi) - p_1(a,b,\varphi))^3}{3 \sin^2 \varphi \cos^2 \varphi} \\ &= \frac{4ab^2}{\cos \varphi} - \frac{4b^3 \sin \varphi}{3 \cos^2 \varphi}, \end{aligned}$$

which results in



$$\begin{aligned}
K(a, b) &= 4 a b^2 \int_0^{x(a, b)} \frac{d(\sin \varphi)}{1 - \sin^2 \varphi} + \frac{4 b^3}{3} \int_0^{x(a, b)} \frac{d(\cos \varphi)}{\cos^2 \varphi} \\
&= 4 a b^2 \int_0^{\sin(x(a, b))} \frac{dz}{1 - z^2} - \frac{4 b^3}{3} \int_{\cos(x(a, b))}^1 \frac{dz}{z^2} \\
&= 2 a b^2 \log \left( \frac{1 + \sin(x(a, b))}{1 - \sin(x(a, b))} \right) - \frac{4 b^3}{3} \left( \frac{1}{\cos(x(a, b))} - 1 \right) \\
&= 2 a b^2 \log \left( \frac{\sqrt{a^2 + b^2} + a}{\sqrt{a^2 + b^2} - a} \right) - \frac{4 b^2}{3} (\sqrt{a^2 + b^2} - b) = \frac{4}{3} I(a, b) .
\end{aligned}$$

Here we have used the relations

$$\int_0^x \frac{dz}{1 - z^2} = \frac{1}{2} \log \frac{1+x}{1-x} \quad \text{for } 0 \leq x < 1 , \quad (7.2)$$

$$\sin(x(a, b)) = \frac{\tan(x(a, b))}{\sqrt{1 + \tan^2(x(a, b))}} = \frac{a}{\sqrt{a^2 + b^2}} , \quad (7.3)$$

$$\cos(x(a, b)) = \frac{1}{\sqrt{1 + \tan^2(x(a, b))}} = \frac{b}{\sqrt{a^2 + b^2}} . \quad (7.4)$$

Next we treat the case (II). For fixed  $\varphi \in [x(a, b), \pi/2]$ , there exist a unique  $p_2 = p_2(a, b, \varphi) \in [0, p^*]$  being equal to the distance of the line with orientation angle  $\varphi$  containing the vertex  $(-a, b)$ , i.e.  $p_2 = -a \cos \varphi + b \sin \varphi$ . The end-points of the chord  $g(p, \varphi) \cap R_{a, b}$  are  $\{(-a, y_1), (a, y_2)\}$  resp.  $\{(x_1, b), (a, y_2)\}$  for  $0 \leq p \leq p_2$  resp.  $p_2 \leq p \leq p^*$ , where

$$-a \cos \varphi + y_1 \sin \varphi = p , \quad a \cos \varphi + y_2 \sin \varphi = p , \quad x_1 \cos \varphi + b \sin \varphi = p$$

so that, for  $0 \leq p \leq p_2(a, b, \varphi)$ , we get

$$(2a)^2 + (y_2 - y_1)^2 = (2a)^2 + \left( \frac{2a \cos \varphi}{\sin \varphi} \right)^2 = 4a^2 (1 + \cot^2 \varphi) = \frac{4a^2}{\sin^2 \varphi}$$

and, for  $p_2(a, b, \varphi) \leq p \leq p^*(a, b, \varphi)$ ,

$$(x_1 - a)^2 + (b - y_2)^2 = \left( \frac{p - b \sin \varphi}{\cos \varphi} - a \right)^2 + \left( \frac{p - a \cos \varphi}{\sin \varphi} - b \right)^2 = \left( \frac{p^*(a, b, \varphi) - p}{\sin \varphi \cos \varphi} \right)^2 .$$

In analogy to case (I) we have to compute the double integral

$$L(a, b) = \int_{x(a, b)}^{\pi/2} \int_0^{p^*(a, b, \varphi)} (\nu_1(g(p, \varphi) \cap R_{a, b}))^2 dp d\varphi .$$

We again treat the inner integral first:

$$\begin{aligned} \int_0^{p^*(a, b, \varphi)} (\nu_1(g(p, \varphi) \cap R_{a, b}))^2 dp &= \int_0^{p_2(a, b, \varphi)} \frac{4a^2}{\sin^2 \varphi} dp + \int_{p_2(a, b, \varphi)}^{p^*(a, b, \varphi)} \left( \frac{p^*(a, b, \varphi) - p}{\sin \varphi \cos \varphi} \right)^2 dp \\ &= \frac{4a^2 p_2(a, b, \varphi)}{\sin^2 \varphi} + \frac{(p^*(a, b, \varphi) - p_2(a, b, \varphi))^3}{3 \sin^2 \varphi \cos^2 \varphi} \\ &= \frac{4a^2 b}{\sin \varphi} - \frac{4a^3 \cos \varphi}{3 \sin^2 \varphi} , \end{aligned}$$

so that we can proceed with

$$\begin{aligned} L(a, b) &= -4a^2 b \int_{x(a, b)}^{\pi/2} \frac{d(\cos \varphi)}{1 - \cos^2 \varphi} - \frac{4a^3}{3} \int_{x(a, b)}^{\pi/2} \frac{d(\sin \varphi)}{\sin^2 \varphi} \\ &= 4a^2 b \int_0^{\cos(x(a, b))} \frac{dz}{1 - z^2} - \frac{4a^3}{3} \int_{\sin(x(a, b))}^1 \frac{dz}{z^2} \\ &= 2a^2 b \log \left( \frac{1 + \cos(x(a, b))}{1 - \cos(x(a, b))} \right) - \frac{4a^3}{3} \left( \frac{1}{\sin(x(a, b))} - 1 \right) \\ &= 2a^2 b \log \left( \frac{\sqrt{a^2 + b^2} + b}{\sqrt{a^2 + b^2} - b} \right) - \frac{4a^2}{3} (\sqrt{a^2 + b^2} - a) = \frac{4}{3} I(b, a) . \end{aligned}$$

Finally, it remains to summarize the above integrals which confirms (6.3) as follows

$$\begin{aligned} J_1^{(2)}(R_{a, b}) &= \frac{1}{\pi} \int_0^\pi \int_{-p^*(a, b, \varphi)}^{p^*(a, b, \varphi)} (\nu_1(g(p, \varphi) \cap R_{a, b}))^2 dp d\varphi \\ &= \frac{4}{\pi} \int_0^{\pi/2} \int_0^{p^*(a, b, \varphi)} (\nu_1(g(p, \varphi) \cap R_{a, b}))^2 dp d\varphi \\ &= \frac{4}{\pi} (K(a, b) + L(a, b)) = \frac{16}{3\pi} (I(a, b) + I(b, a)) . \quad \square \end{aligned}$$

As by-product we may deduce from (6.4) for  $K = R_{a,b}$  the inequality

$$I(a, b) + I(b, a) \leq 8 a b \sqrt{\frac{a b}{\pi}} \quad \text{for all } a, b > 0 .$$

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